

# On common solutions of Mathisson equations under different conditions

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## **Abstract.**

In the context of investigations of possible highly relativistic motions of a spinning particle in the gravitational field, which can be described by the Mathisson equations under different supplementary condition, we analyze the circular orbits in a Schwarzschild field. The very orbits most clearly demonstrate the effect of the gravitational spin-orbit interaction on the particle's motion. It is shown that the Mathisson equations under the Frenkel-Mathisson and Tulczyjew-Dixon conditions have the common solutions describing the highly relativistic circular orbits in the region  $r = 3M(1 + \delta)$ ,  $|\delta| \ll 1$ . These orbits essentially differ from the geodesic circular orbits in the same region, particularly by the value of the particle's energy on the corresponding orbits.

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## 1. Introduction

The investigations of the behavior of a spinning particle in the gravitational field are based on the analysis of solutions of the Mathisson equations [1] and the general relativistic Dirac equation [2]. In the first case we deal with the description of the world line and trajectory of a macroscopic body with the inner rotation, and in the second case with the analysis of the properties of the wave function of a quantum particle. Under certain circumstances, in the quasi classical approximation, the Mathisson equations follow from the Dirac equation [3]. In this context, it is important that new solutions of the Mathisson equations stimulate the investigation of the corresponding quantum states [4].

The Mathisson-Papapetrou equations can be written in the form [1]

$$\frac{D}{ds} \left( mu^\lambda + u_\mu \frac{DS^{\lambda\mu}}{ds} \right) = -\frac{1}{2} u^\pi S^{\rho\sigma} R_{\pi\rho\sigma}^\lambda, \quad (1)$$

$$\frac{DS^{\mu\nu}}{ds} + u^\mu u_\sigma \frac{DS^{\nu\sigma}}{ds} - u^\nu u_\sigma \frac{DS^{\mu\sigma}}{ds} = 0, \quad (2)$$

where  $u^\lambda \equiv dx^\lambda/ds$  is the 4-velocity of a spinning particle,  $S^{\mu\nu}$  is the tensor of spin,  $m$  and  $D/ds$  are, respectively, the mass and the covariant derivative;  $R_{\pi\rho\sigma}^\lambda$  is the Riemann curvature tensor of the spacetime. (Greek indices run 1, 2, 3, 4 and Latin indices 1, 2, 3.) After [1] equations (1), (2) were derived by many authors using different methods [5]. Often these equations are named as the Mathisson-Papapetrou-Dixon equations.

An essential circumstance which complicates the study of physical results following from equations (1), (2) is the necessity to supplement these equations by the certain condition for the description of motions of the particle's center of mass. However, in the relativistic mechanics in contrast to the Newtonian mechanics, such condition is not determined uniquely. Two conditions are usually imposed:

$$S^{\lambda\nu} u_\nu = 0, \quad (3)$$

or

$$S^{\lambda\nu} P_\nu = 0, \quad (4)$$

where

$$P^\nu = mu^\nu + u_\lambda \frac{DS^{\nu\lambda}}{ds}. \quad (5)$$

Relations (3) and (4) are called the Frenkel-Mathisson [1, 7] and Tulczyjew-Dixon [8, 9] conditions correspondingly. (After [10] relation (3) is often called the Pirani condition.)

Generally, the solutions of equations (1), (2) at conditions (3), (4) are different. For example, in the Minkowski spacetime equations (1)–(3) have both the solutions which describe the straight worldlines and the solutions describing the oscillatory (helical) worldlines [11, 12]. Whereas equations (1), (2), (4) do not admit the oscillatory solutions. (The interpretation of these unusual solutions was proposed by Möller in the terms of the proper and non-proper centers of mass [13].) Nevertheless, there are the common

solutions of equations (1), (2) under conditions (3) and (4) which describe the motions of the proper center of mass. If the gravitational field is present and is considered in the post-Newtonian approximation, the solutions of equations (1), (2) at (3) and (4) are close with high accuracy [14]. More generally, the similar situation takes place if the effect of the particle's spin can be described by the power in spin corrections to the corresponding expressions for the geodesic motions [15].

Special analysis must be carry out in the case when the motion of the proper center of mass cannot be described by the small corrections in spin to the geodesic motion. Such a conclusion follows, for example, from [4,16–18] after the analysis of solutions of equations (1), (2), (3), where it is shown that for some highly relativistic motions of a spinning particle relative to a Schwarzschild mass the world line of this particle can differ essentially from the geodesic world line. Therefore, the purpose of the present paper is to investigate the non-trivial common solutions of equations (1), (2) under conditions (3) and (4) in a Schwarzschild field.

It is known that the deviation of the spinning particle motion from the geodesic one in a Schwarzschild field is caused by the gravitational spin-orbit interaction [19] (the case of the highly relativistic motions is considered in [4, 16]). As a result, the maximal effect of the particle's spin on its trajectory takes place for circular or close to circular motions. Therefore, in the following we shall study such solutions of equations (1), (2), (4) which describe the circular orbits of a spinning particle in the Schwarzschild field. It is important that these solutions can be obtained in the analytic form.

Different trajectories of a spinning particles following from equations (1), (2), (4) in the Schwarzschild and Kerr fields were investigated in [20–24] depending on the concrete values of the integrals of energy and angular momentum. Particularly, in [21] the method of the effective potential was used for the classification of different orbits, including the chaotic motions. In contrast to these investigations, we shall consider the circular orbits only, however, without the restriction by the small in spin perturbations. That is, the object of our investigations are the strict equations (1), (2), (4), and their explicit solutions we shall find without initial using the integrals of energy and angular momentum. The values of these integrals will be estimated after obtaining the explicit expressions for the quantities which determines the energy and angular momentum.

We stress that the necessary condition for a spinning test particle [19]

$$\frac{|S_0|}{mr} \equiv \varepsilon \ll 1 \quad (6)$$

will be taken into account in our consideration, where  $|S_0|$  is the absolute value of spin,  $r$  is the radial coordinate.

This paper is organized as follows. In Section 2 the general relations following from equations (1), (2), (4) for any metric are written. The main calculations concerning the analysis of possible solutions describing the circular orbits in a Schwarzschild field are presented in Section 3. The region of existence of the highly relativistic circular orbits and the particle's energy on these orbits are studied in Sections 4 and 5. We conclude in Section 6.

## 2. Basic relations following from Mathisson equations under condition (4)

Usually equations (1), (2) at condition (4) are written as

$$\frac{DP^\lambda}{ds} = -\frac{1}{2}u^\pi S^{\rho\sigma} R_{\pi\rho\sigma}^\lambda, \quad (7)$$

$$\frac{DS^{\lambda\nu}}{ds} = P^\lambda u^\nu - P^\nu u^\lambda. \quad (8)$$

The quantity

$$\mu = \sqrt{P_\lambda P^\lambda} \quad (9)$$

is the mass of a particle, and according to (7), (8) at condition (4),  $\mu$  is the integral of motion:

$$\frac{d\mu}{ds} = 0. \quad (10)$$

(Under condition (3) the constant quantity is  $m$  in equation (1)).

There is connection between the 4-velocity  $u^\lambda$  and the 4-momentum  $P^\lambda$  [21]

$$u^\lambda = N \left[ \frac{P^\lambda}{\mu} + \frac{1}{2\mu^3 \Delta} S^{\lambda\nu} P^\pi R_{\nu\pi\rho\sigma} S^{\rho\sigma} \right], \quad (11)$$

where

$$\Delta = 1 + \frac{1}{4\mu^2} R_{\lambda\pi\rho\sigma} S^{\lambda\pi} S^{\rho\sigma}, \quad (12)$$

$$N = \left[ 1 + \frac{1}{4\Delta^2 \mu^4} S_{\pi\nu} P_\lambda S_{\rho\sigma} R^{\nu\lambda\rho\sigma} S^{\pi\alpha} P^\beta S^{\gamma\delta} R_{\alpha\beta\gamma\delta} \right]^{-1/2}. \quad (13)$$

It follows from (11) that  $P_\lambda P^\lambda = \mu^2$ . The integral of motion is the value of spin  $S_0$ :

$$S_0^2 \equiv \frac{1}{2} S_{\lambda\nu} S^{\lambda\nu}. \quad (14)$$

## 3. Equations following from (7), (8) for equatorial circular orbits in a Schwarzschild field

We use the Schwarzschild metric in the standard coordinates  $x^1 = r$ ,  $x^2 = \theta$ ,  $x^3 = \varphi$ ,  $x^4 = t$  with the non-zero components of the metric tensor  $g_{\lambda\nu}$ :

$$g_{11} = - \left( 1 - \frac{2M}{r} \right)^{-1}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta, \quad g_{44} = 1 - \frac{2M}{r}. \quad (15)$$

Let us consider the relations following from equations (7), (8) under condition (4) for the equatorial circular motions  $\theta = \pi/2$  of a spinning particle with the constant angular velocity around the Schwarzschild mass. That is, in (7), (8) we put

$$u^1 = 0, \quad u^2 = 0, \quad u^3 = \text{const} \neq 0, \quad u^4 = \text{const} \neq 0, \quad (16)$$

$$S^{12} = 0, \quad S^{23} = 0, \quad S^{13} = \text{const} \neq 0, \quad (17)$$

without any *a priori* restriction on  $P_\lambda$ . According to (4), (16), (17) we have

$$S^{14} = -\frac{P_3}{P_4}S^{13}, \quad S^{24} = 0, \quad S^{34} = \frac{P_1}{P_4}S^{13}. \quad (18)$$

Then for the derivatives  $DS^{\lambda\nu}/ds$  we find

$$\frac{DS^{12}}{ds} = 0, \quad \frac{DS^{13}}{ds} = -\frac{P_1}{P_4}\Gamma_{44}^1 u^4 S^{13}, \quad \frac{DS^{23}}{ds} = 0, \quad (19)$$

where  $\Gamma_{44}^1$  is the Christoffel symbol calculated by metric (15). After (16)–(19) we obtain from (8) the two non-trivial relations:

$$-\frac{P_1}{P_4}\Gamma_{44}^1 u^4 S^{13} = P^1 u^3, \quad (20)$$

$$P^2 u^3 = 0. \quad (21)$$

Because  $u^3 = d\varphi/ds \neq 0$ , by (21) it is necessary

$$P^2 = 0. \quad (22)$$

Relation (20) is fulfilled in two cases:

$$P^1 = 0, \quad (23)$$

or

$$-g_{11}u^4\Gamma_{44}^1 S^{13} = P_4 u^3. \quad (24)$$

First we shall analyze case (23). Then the two first relations (11), with  $\lambda = 1, 2$ , are satisfied identically, and others take the form

$$u^3 = N \frac{P^3}{\mu} \left( 1 - \frac{3S_0^2}{\mu^4 \Delta} P_4 P^4 R^{13}_{13} \right), \quad (25)$$

$$u^4 = N \frac{P^4}{\mu} \left( 1 + \frac{3S_0^2}{\mu^4 \Delta} P_3 P^3 R^{13}_{13} \right), \quad (26)$$

where

$$\Delta = 1 + \frac{S_0^2}{\mu^2} R^{13}_{13} (1 - 3P_3 P^3), \quad (27)$$

$$N = \left( 1 + \frac{A}{4\Delta^2 \mu^4} \right)^{-1/2}, \quad (28)$$

$$A \equiv 36S_0^4 R_{1313} R^{1313} P_3 P^3 P_4 P^4. \quad (29)$$

To obtain (25), (26) we used the expressions

$$(S^{13})^2 = S_0^2 P_4 P^4 g^{11} g^{33}, \quad (S^{14})^2 = S_0^2 P_3 P^3 g^{11} g^{44} \quad (30)$$

following from (4), (14).

Now we shall consider equations (7) at conditions (16)–(19), (22), (23). The first equation of set (7), with  $\lambda = 1$ , takes the form

$$\Gamma_{33}^1 u^3 P^3 + \Gamma_{44}^1 u^4 P^4 = -g_{44}^2 \sqrt{g^{11} g^{33} g^{44}} \frac{S_0}{\mu} (2P^3 u^4 + P^4 u^3). \quad (31)$$

The second equation of (7), with  $\lambda = 2$ , is satisfied identically. From the third and fourth equations of (7) it is easy to obtain correspondingly

$$P^3 = \text{const}, \quad P^4 = \text{const}. \quad (32)$$

So, all equations (7), (8) will be satisfied if the four constant values  $u^3, u^4, P^3, P^4$ , with relations (25), (26), satisfy equation (31).

#### 4. Region of existence of equatorial circular orbits

From the algebraic relations (25), (26), (31) it is not difficult to obtain the second order equation for the value

$$x \equiv P_3 P^3. \quad (33)$$

This equation is:

$$\begin{aligned} & x^2 \left[ \left( 1 - \frac{3M}{r} - \frac{3M^2}{r^2} \varepsilon^2 \right)^2 - 9\varepsilon^2 \frac{M^2}{r^2} \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{4M}{r} \varepsilon^2 \right) \right] \\ & + \mu^2 x \left[ \frac{2M}{r} \left( 1 - \frac{3M}{r} - \frac{3M^2}{r^2} \varepsilon^2 \right)^2 \left( 1 + \frac{M}{r} \varepsilon^2 \right) + 9\varepsilon^2 \frac{M^2}{r^2} \left( 1 - \frac{2M}{r} \right) \left( 1 + \frac{4M}{r} \varepsilon^2 \right) \right] \\ & + \mu^4 \frac{M^2}{r^2} \left( 1 + \frac{2M}{r} \varepsilon^2 \right) = 0, \end{aligned} \quad (34)$$

where

$$\varepsilon^2 \equiv \frac{S_0^2}{\mu^2 r^2} \quad (35)$$

and according to condition (6) it is necessary  $\varepsilon^2 \ll 1$ . In the trivial case of a spinless particle ( $\varepsilon = 0$ ) from (34) we have

$$x = -\mu^2 \frac{M}{r} \left( 1 - \frac{3M}{r} \right)^{-1}. \quad (36)$$

As a result, according to (33),

$$P^3 = \pm \mu \sqrt{\frac{M}{r^3}} \left( 1 - \frac{3M}{r} \right)^{-1/2}. \quad (37)$$

Equation (37) coincides with the known expression for the 4-momentum of a spinless particle with the mass  $\mu$  which follows directly from the geodesic equations for the circular orbits. Particularly, equation (37) shows that such orbits in a Schwarzschild field exist only at  $r > 3M$ .

It is easy to see that in the case

$$1 - \frac{3M}{r} \gg \frac{M^2}{r^2} \varepsilon^2, \quad (38)$$

*i.e.*, if  $r$  is not close to  $3M$ , it follows from (34) the expression for  $P^3$  which differs from (37) only by the small correction of order  $\varepsilon^2$ .

The especial case takes place if  $r$  is equal or close to  $3M$ . Particularly, at

$$r = 3M, \quad \epsilon \neq 0 \quad (39)$$

we obtain from (34) the single negative root

$$x = -\frac{1}{\sqrt{3}\mu^2\epsilon}, \quad (40)$$

where without loss in generality we put

$$\epsilon \equiv \frac{S_0}{\mu r}, \quad (41)$$

*i.e.*,  $S_0 > 0$ . (According to our choice of the metric signature, the value  $x$  in (34) is negative). By (33), (40) we have  $P^3 = \pm 3^{-5/4}\epsilon^{-1/2}\mu M^{-1}$ . However, it is easy to check that all relations (25), (26), (31) are satisfied only if

$$P^3 = -\frac{\mu}{3^{5/4}M\sqrt{\epsilon}}. \quad (42)$$

Taking into account relations (25)-(29), with the accuracy to  $\epsilon$  we write  $u^3 = P^3/\mu$ , then according to (42) for the angular velocity we have

$$u^3 = -\frac{1}{3^{5/4}M\sqrt{\epsilon}}. \quad (43)$$

If  $r$  is not equal to  $3M$ , however is close to this value:

$$1 - \frac{3M}{r} = \delta, \quad |\delta| \ll \epsilon, \quad (44)$$

then instead of (42), (43) we obtain from (34)

$$P^3 = -\frac{\mu}{3^{5/4}M\sqrt{\epsilon}} \left( 1 - \frac{\sqrt{3}\delta}{2\epsilon} \right), \quad (45)$$

$$u^3 = -\frac{1}{3^{5/4}M\sqrt{\epsilon}} \left( 1 - \frac{\sqrt{3}\delta}{2\epsilon} \right). \quad (46)$$

We stress that here  $\delta$  may be both positive and negative. That is, equation (34) admit the circular orbits within the small neighborhood of  $r = 3M$ , if  $|\delta| \ll \epsilon$ , both at  $r > 3M$  and  $r < 3M$ .

Now we can compare expressions (43), (46) with the corresponding expressions for  $u^3$  obtained in [17] from equations (1), (2) at condition (3). It is easy to see that these expressions coincide (it is necessary to take into account that in [17] the notation for  $\epsilon$  was used  $\epsilon \equiv |S_0|/Mm$ , in contrast to (6)). So, we conclude that the existence of the circular orbits of a spinning particle in the small neighborhood of the value  $r = 3M$  is the common result of equations (1), (2) under conditions (3) and (4).

Concerning relation (24) it is not difficult to check that in this case equations (7), (8) are not compatible. That is, case (23) is the single possible one.

## 5. Values of energy and orbital momentum

Let us estimate the energy  $E$  and orbital momentum  $L$  of a spinning particle on the above considered circular orbits. The expressions of these values for the equatorial motions in a Schwarzschild field are [25]

$$E = P_4 + \frac{1}{2}g_{44,1}S^{14}, \quad (47)$$

$$L = -P_3 - \frac{1}{2}g_{33,1}S^{13}. \quad (48)$$

According to relations (6), (9), (30), (45) we have

$$|g_{44,1}S^{14}| \ll P_4, \quad (49)$$

therefore approximately we write

$$E = \mu \frac{3^{-3/4}}{\sqrt{\varepsilon}} \left( 1 - \frac{\sqrt{3}\delta}{2\varepsilon} \right). \quad (50)$$

Let us compare this value of the energy of a spinning particle on the circular orbit from the small neighborhood of  $r = 3M$  with the value of energy of a spinless particle on the geodesic circular orbit of  $r = 3M(1 + \delta)$ ,  $0 < \delta \ll 1$ . By (37) we have

$$E_{geod} = \frac{\mu}{3\sqrt{\delta}}. \quad (51)$$

It follows from (50), (51) that

$$E^2/\mu^2 \gg 1, \quad E_{geod}^2/\mu^2 \gg 1, \quad (52)$$

*i.e.*, these values are highly relativistic. At the same time, according to (44) we have

$$E^2/E_{geod}^2 = \frac{\sqrt{3}\delta}{\varepsilon} \ll 1, \quad (53)$$

that is the energies of the spinning and spinless particles on the circular orbits from the small neighborhood of  $r = 3M$  differ to a great extent. It is not strange because different physical reasons cause the existence of these orbits: the geodesic circular orbits exist due to the gravitational attraction, whereas the circular orbits of a spinning particle are determined by the common action of the gravitational attraction and the gravitational spin-orbit repulsion. In addition, we stress that from the point of view of the comoving observer the acceleration of a spinning particle relative to a spinless particle is of order  $M/r^2$ , *i.e.*, is considerable [4]. Therefore, the circular orbits of a spinning particle from the small neighborhood of  $r = 3M$  are essentially non-geodesic orbits.

It is not difficult to check that the similar conclusions follow from the analysis of the orbital momentum (48).



## 6. Conclusions

So, Mathisson equations (1), (2) at Tulczyjew-Dixon conditions (4) have the solutions which describe highly relativistic circular orbits of a spinning particle in the small neighborhood of the value  $r = 3M$  in a Schwarzschild field, both for  $r > 3M$  and  $r < 3M$ . The same orbits are allowed by the Mathisson equations at Frenkel-Mathisson condition (3), that was shown in [17] while investigating possible synchrotron radiation of a charged spinning particle in a Schwarzschild field. It is important that the dynamics of a spinning particle on these orbits essentially differ from the dynamics of a spinless particle on the geodesic circular highly relativistic orbits which exist in the small neighborhood of  $r = 3M$ , at  $r > 3M$ .

The question arise: do equations (1), (2) under condition (4) allow other types of orbits which are not circular and differ considerable from the corresponding geodesic orbits (naturally, when condition (6) is satisfied)? To answer this question it is necessary to carry out more complex calculation. We point out that the solutions of equations (1), (2), (3) in a Schwarzschild field, which describe the essentially non-geodesic non-circular orbits of the proper center of mass of a spinning particle, were considered in [18].

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